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Roles of surface tension and Reynolds stresses on the finite amplitude stability of a parallel flow with a free surface

By S. P. LIN

Mechanical Engineering Department, Clarkson College of Technology, Potsdam, N.Y.

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Subcritically stable motion of long gravity waves of finite amplitude in a liquid layer flowing down an inclined plane is shown to be impossible. However, supercritically stable wave régimes for such flows are found and curves of constant wave amplitude in such régimes are obtained. The mechanism of non-linear stability is investigated by considering the energy transfer between the mean flow and the disturbances. The results obtained show that the mechanism of stability in a parallel flow with a free surface is quite different from that in a parallel flow without a free surface.

1. Introduction

In a previous paper (henceforth referred to as [I]) Lin (1969) obtained a closed form solution to the problem of finite amplitude stability of a liquid layer flowing down an inclined plane. It was shown that in the neighbourhood of the neutral curve obtained by Benjamin (1957) and Yih (1963), an exponentially growing infinitesimal disturbance may develop into supercritically stable wave motion of small but finite amplitude observed by Kapitza (1949) and Binnie (1957). However, it was not obvious from the obtained closed form solution whether such a film flow will exhibit subcritical instability. Nor could the limit of supercritically stable wave motion be readily revealed from the complicated expression for the second Landau coefficient. In the present paper, numerical computations have been carried out to determine such a limit for falling liquid films of different angles of inclination and surface tensions. In supercritically stable régimes, curves of constant wave amplitude have been obtained. The numerical results also show that all small but finite disturbances are damped in subcritical regions. Following the line of Stuart's (1960) analysis, the mechanism of non-linear stability is investigated from the point of view of energy transfer between the mean flow and disturbances. The results of analysis are then compared with Reynolds & Potter's (1967) results for a parallel flow without a free surface. The comparison shows striking differences between these two cases.

2. Supercritical wave motion

Stuart (1960) has derived from the Navier–Stokes equation the following expression which governs the non-linear growth of a periodic finite disturbance in a parallel flow without a free surface

$$\frac{1}{2}\frac{dA^2}{dt} = \alpha c_i A^2 + a^{[2]} A^4 + O(A^6), \tag{1}$$

where A is the disturbance amplitude, α the wave-number, $a^{[2]}$ the often called Landau's second coefficient and c_i is the imaginary part of the eigenvalue obtained in the linearized stability theory. It will be shown in the next section that the above equation still governs the growth of disturbances in a parallel flow *with* a free surface. $a^{[2]}$ in the above equation, being defined as $\alpha c_i^{[2]}$, is given by equation (32) in [I] and c_i is given by Benjamin (1957) and Yih (1963) to be

$$c_i = a[6R/5 - (\cot\beta + \alpha^2\sigma/3)], \tag{2}$$

where β is the angle of inclination of the plane, R the Reynolds number and σ a surface tension parameter defined respectively by

$$R = \overline{U}_a d/\nu, \quad \sigma = RS$$

with $S = T/\rho d\overline{U}_a^2$ and $\overline{U}_a = gd^2 \sin\beta/3\nu$. In the above equations, $\overline{U}_a$ is the average velocity in the film flow, ν the kinematic viscosity, T the surface tension, ρ the density of the liquid, g the gravitational acceleration and d is the thickness of the film. For given values of σ and β, neutral curves along which c_i vanishes have been calculated from (2) and are plotted in figures 1 and 2. Each of these neutral curves divides the α, R plane into two regions. c_i is positive in the region below each neutral curve and is negative in the rest of the α, R plane. In the very near vicinity of neutral curves, the first term on the right side of (1) dominates and disturbances will grow or decay exponentially according to whether $c_i > 0$ or $c_i < 0$. However, this amplitude will immediately make the second term on the right side of (1) significant, and consequently the sign of $a^{[2]}$ plays an important role in the non-linear development of the initial disturbance. Curves along which $a^{[2]}$ vanishes are calculated from (32) given in [I] and are plotted in figures 1 and 2. $a^{[2]}$ is positive below each of these curves and is negative in the rest of the α, R plane. (The symbol Q appearing in (32) was not given in [I] and should be defined as $Q = 3\cot\beta/R + \alpha^2 S$.) Figures 1 and 2 show, at least for calculated values of σ and β, that $a^{[2]}$ is negative in the region of negative c_i. It follows from (1) that no subcritically stable wave motion is possible since all finite but small disturbances of long wavelength are damped in the region of $c_i < 0$. On the other hand, $a^{[2]} < 0$ and $c_i > 0$ in the region defined by each pair of neutral curves and curves of zero Landau second coefficient as shown in figures 1 and 2. Thus, within this region a supercritically stable wave motion is possible for given values of β and σ. In these wave régimes, curves of constant amplitude are calculated from (1). It is observed that R in (1), with $a^{[2]}$ given by (32) in [I], appears only in forms of products αR and RS. This allows us to assign values of A, σ and α in (1) and then solve the resulting equation in αR by Newton–Raphson iteration. Having obtained αR for which $dA^2/dt = 0$ for given A, α and σ, we can obtain the corresponding R by a simple division. The results are plotted in figures 3 to 5. Some typical results are also given in table 1. If the degree of stability is defined as the increment in the Reynolds number for a given increment in the wave amplitude, then figures 3 to 5 show that for extremely long waves the degree of stability is very small. As we reduce the wavelength slightly, the degree of stability is seen to increase rapidly until it reaches the maximum. If the wavelength is reduced further, the same figures show a gradual decrease in the degree of stability. It is clear that there is a maximum

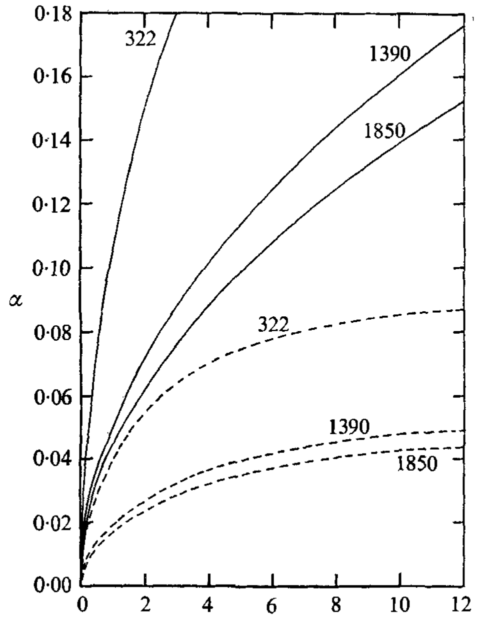

FIGURE 1. Stability curves, $\beta = 90°$: ——, $c_i = 0$; - - - -, $a^{[2]} = 0$; the numbers appearing in the figure are the values of σ.

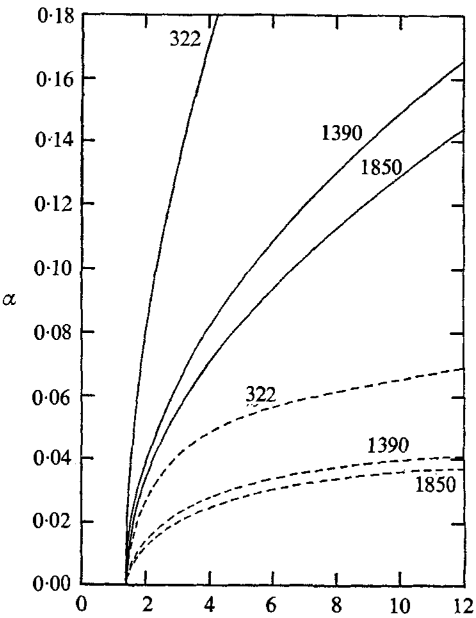

FIGURE 2. Stability curves, $\beta = 30°$; ——, $c_i = 0$; - - - -, $a^{[2]} = 0$; the numbers appearing in the figure are the values of σ.

of $a^{[2]}$ for a given A in the range of α of interest. However, it is not clear what causes this peculiar phenomenon. For each curve of constant wave amplitude shown in figures 3–5, corresponding values of wave speed have also been calculated from (35) and (33) in [I]. Some of the representative results are given together with other relevant results in table 1. Figure 6 shows the dependence of wave speed on the amplitude and wavelength. For a given wave amplitude, the wave speed seems to increase with the wavelength. For a given wavelength, the wave speed is seen to increase with amplitude. Thus, the supercritically stable waves we have found seem to possess the characteristics of finite amplitude gravity waves (Stokes 1847) rather than capillary waves the speed of which is known to decrease as the amplitude increases (Crapper 1957). In order to gain a better understanding of the roles of surface tension, viscosity and gravitational potential in the non-linear stability which is indicated by the above computational study, we shall consider the energy transfer between the mean flow and disturbances in a falling liquid layer.

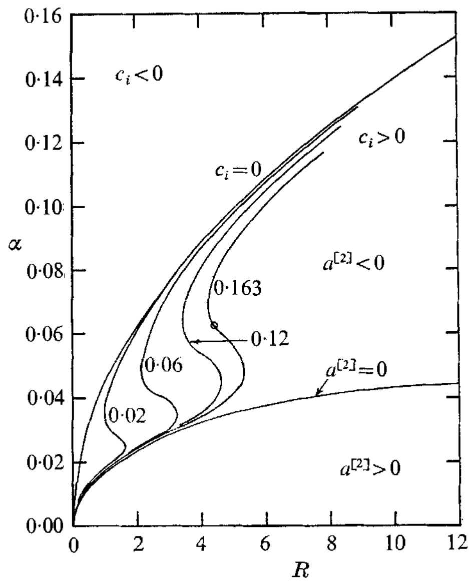

FIGURE 3. Supercritically stable wave régime, $\beta = 90°$, $\sigma = 1850{\cdot}0$. ○, Binnie's experiment, $\alpha = 0{\cdot}0615$, $R = 4{\cdot}4$. The value of wave amplitude is given on the right of each curve.

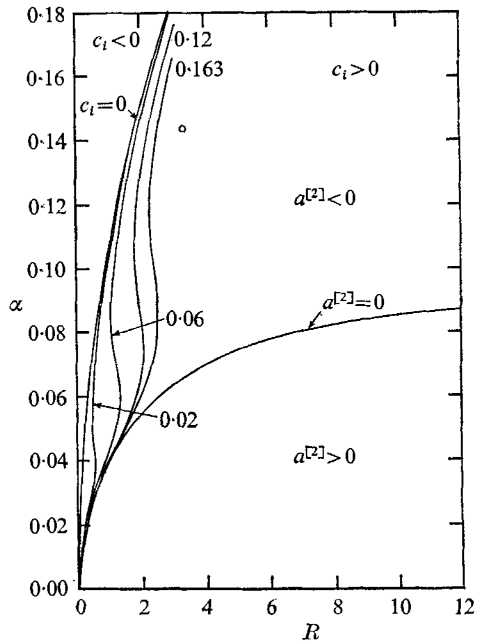

FIGURE 4. Supercritically stable wave régime, $\beta = 90°$, $\sigma = 322{\cdot}0$. ○, Kapitza's experiment, $\alpha = 0{\cdot}144$, $R = 3{\cdot}35$, $A = 0{\cdot}17$. The value of wave amplitude is given on the right of each curve.

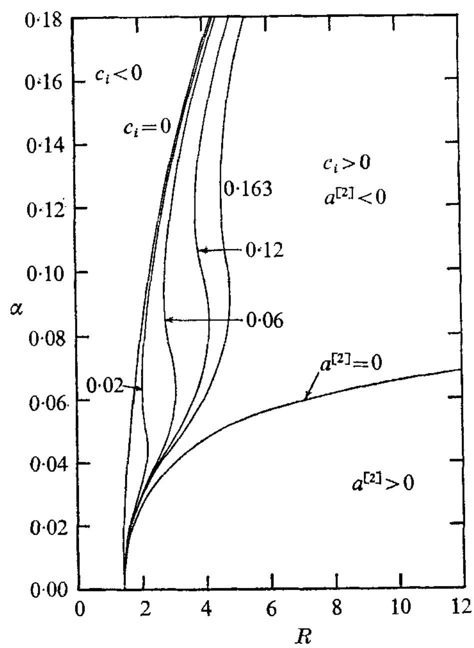

FIGURE 5. Supercritically stable wave régime, $\beta = 30°$, $\sigma = 322{\cdot}0$. The value of constant amplitude is given on the right of each curve.

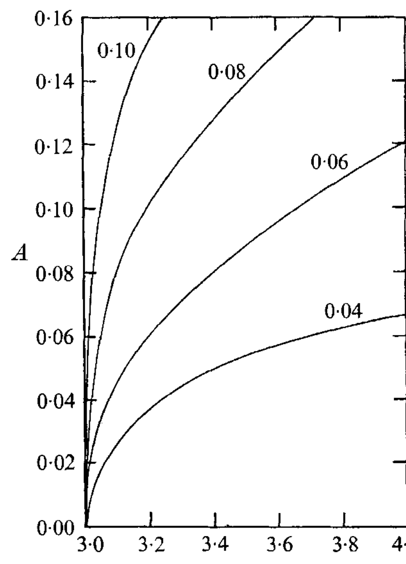

FIGURE 6. Dependence of wave speed on the amplitude and wavelength. The numbers appear in the figure are wave-numbers. $\beta = 90°$, $\sigma = 322{\cdot}0$.

α	R	c	k_1	k_2	k_3	$2a^{[2]}$	c_i
0·0200	0·2477	5·9816	−0·0000	4·9241	−4·9817	−0·0576	0·0052
0·0250	0·3838	4·8929	−0·0000	1·6096	−1·7463	−0·1367	0·0098
0·0300	0·5435	4·2960	−0·0000	0·6356	−0·9138	−0·2782	0·0167
0·0350	0·7194	3·9311	−0·0000	−0·6239	0·1255	−0·4984	0·0256
0·0370	0·7929	3·8237	−0·0000	−2·2182	1·6060	−0·6121	0·0298
0·0400	0·8995	3·6895	−0·0000	−2·8025	1·9956	−0·8069	0·0363
0·0450	1·0648	3·5176	−0·0000	−3·9958	2·8022	−1·1936	0·0477
0·0600	1·2963	3·2003	−0·0000	−7·5042	5·1658	−2·3383	0·0702
0·0800	1·0934	3·0446	−0·0000	−6·4129	4·1896	−2·2233	0·0500
0·1000	1·1758	3·0135	−0·0000	−4·4505	2·5749	−1·8756	0·0338
0·1200	1·4584	3·0039	−0·0000	−3·3320	1·6962	−1·6358	0·0245
0·1400	1·8675	3·0004	−0·0000	−2·6549	1·1596	−1·4953	0·0192
0·1600	2·3767	3·0000	−0·0000	−2·2136	0·7293	−1·4843	0·0167
0·1800	2·9728	3·0017	−0·0000	−1·9115	0·2940	−1·6176	0·0162

TABLE 1. Typical results. $\beta = 90°$, $\sigma = 332·0$, $A = 0·06$.

3. Mechanism of non-linear stability

Consider a layer of liquid flowing down an inclined plane $y = 1$ (cf. figure 1 of [I]). The governing equation of motion is

$$\frac{\partial \mathbf{V}_1}{\partial t}+[(\mathbf{V}_0+\mathbf{V}_1)\,\mathrm{grad}](\mathbf{V}_0+\mathbf{V}_1) = -\mathrm{grad}\,p_1+\frac{1}{R}\nabla^2\mathbf{V}_1, \tag{3}$$

where $\mathbf{V}_0$ is the velocity field of the unperturbed flow, $\mathbf{V}_1$ the velocity disturbance and p_1 is the pressure disturbance. If we split the disturbances in (3) into three parts as

$$\mathbf{V}_1 = \mathbf{V}^0+\mathbf{V}'+\mathbf{V}'', \quad p_1 = p^0+p'+p'',$$

where the superscript ^{0} denotes the non-harmonic part of the disturbance and prime or double prime denotes the harmonic disturbances of odd or even order, and we then form dot products of all terms in (3) with $\mathbf{V}'$ and then integrate from the free surface $y = \eta(x,t)$ to the plane $y = 1$ over a wavelength, we have

$$\begin{aligned}\frac{\partial}{\partial t}\int\int_2^1 (u'^2+v'^2)\,dx\,dy &= \int\int(-u'v')\frac{\partial \bar{u}}{\partial y}dx\,dy-\int\left[-p'+\frac{2}{R}\frac{\partial v'}{\partial y}\right]_\eta dx\\ &-\frac{1}{2R}\int\int\left[\left(2\frac{\partial v'}{\partial x}\right)^2+\left(2\frac{\partial v'}{\partial y}\right)^2+2\left(\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right)^2\right]dx\,dy-\int(u'^2-v'^2)\frac{\partial u''}{\partial x}dx\,dy\\ &-\int\int u'v'\left(\frac{\partial u''}{\partial y}+\frac{\partial v''}{\partial x}\right)dx\,dy+\frac{1}{2}\int[v''(u'^2+v'^2)]_\eta dx,\end{aligned} \tag{4}$$

where superscripts ^{0}, prime and double prime have the same representation as before and u or v are used to stand for the x and y components of the velocity disturbances. $\bar{u}$ in (4) is the distorted mean flow consisting of the primary flow and the x component of the non-harmonic part of the velocity disturbances. Equation (4) states that the time rate of change of kinetic energy in odd disturbances is equal to the rate of energy transfer from the mean flow to odd disturbances less the rate of work done by the normal stress to deform the free surface, less the rate of energy dissipation through viscosity, less the energy transfer between odd and even disturbances which is given by the last three integrals in (4). If the free surface is replaced by a rigid plane, the second and the last integrals in (4) with integrands being evaluated at $\eta = 0$ will vanish, and thus Stuart's (1956) result is recovered. It will be seen shortly that the presence of a free surface leads to a significant modification on the mechanism of non-linear stability. Substituting the expressions for the velocity and pressure disturbances given in [I] into (4), we have, after some algebraic manipulation,

$$[1+O(A^2)]\frac{dA^2}{dt} = 2\alpha c_i A^2 + (k_1+k_2+k_3)A^4 + O(A^6), \tag{5}$$

where

$$k_1 = -\frac{4\alpha}{l_0}\iint \operatorname{Re}\phi''^{(0;\,2)} \operatorname{Im}\phi^{(1;\,1)}\bar{\phi}'^{(1;\,1)}dx\,dy,$$

$$k_2 = \frac{2\alpha}{l_0}\iint \operatorname{Im}\,[2\phi'^{(2;\,2)}\bar{\phi}'^{(1;\,1)2} + 2\alpha^2\phi'^{(2;\,2)}\bar{\phi}^{(1;\,1)2} + \bar{\phi}^{(1;\,1)}\bar{\phi}'^{(1;\,1)}\phi''^{(2;\,2)}$$
$$+ 4\alpha^2\bar{\phi}^{(1;\,1)}\bar{\phi}'^{(1;\,1)}\phi^{(2;\,2)}]\,dx\,dy - \frac{2\alpha}{l_0}\int \operatorname{Im}\,[\phi'^{(1;\,1)2}\bar{\phi}^{(2;\,2)} + \alpha^2\bar{\phi}^{(1;\,1)2}\phi^{(2;\,2)}]_\eta^1\,dx,$$

$$k_3 = \frac{\alpha}{l_0}\iint\left\{\frac{2}{\alpha R}\operatorname{Re}\,[\phi'^{(1;\,1)}\bar{\phi}'''^{(1;\,3)} - \alpha^2(\phi'^{(1;\,3)}\phi'^{(1;\,1)} - \phi^{(1;\,1)}\bar{\phi}''^{(1;\,3)})]\right.$$
$$-2\operatorname{Im}\,[\bar{\phi}'^{(1;\,1)}p^{(1;\,3)} + \bar{\phi}'^{(1;\,3)}p^{(1;\,1)} + \bar{\phi}^{(1;\,1)}p'^{(1;\,3)} + \bar{\phi}^{(1;\,3)}p^{(1;\,1)}]$$
$$\left. + 6y\operatorname{Im}\,[\phi^{(1;\,3)}\bar{\phi}'^{(1;\,1)} + \phi^{(1;\,1)}\bar{\phi}'^{(1;\,3)}]\right\}dx\,dy,$$

with

$$l_0 = \iint (|\phi'^{(1;\,1)}|^2 + \alpha^2|\phi^{(1;\,1)}|^2)\,dx\,dy.$$

In the above expressions prime denotes differentiation with respect to y and ϕ's are functions of y already obtained in [I]. From the above development, the mechanism of non-linear stability may be seen to involve the following three competing processes: (*a*) distortion of the mean flow by the Reynolds stress apparent in k_1; (*b*) generation of higher harmonics or amplification of the fundamental through Reynolds stresses depending on whether $k_2 < 0$ or $k_2 > 0$; (*c*) redistribution of the fundamental harmonic arising from (i) energy dissipation through viscosity, (ii) free surface deformation by the normal stress and (iii) the energy transfer from the mean flow to the fundamental; all given in k_3.

For each curve of constant wave amplitude given in figures 3 to 5, corresponding values of k_1, k_2 and k_3 have been calculated from the above equations by the method of numerical quadrature (Ralton & Wilf 1967). Some of the representative results are given in table 1. k_1 is found to be negative but extremely small compared with k_2 or k_3. Thus, process (*a*) mentioned above is insignificant, although it is a stablizing process in the non-linear development of long gravity waves. This situation is quite different from that in a parallel flow without a free surface studied by Reynolds & Potter (1967). Their calculations show that distortion of the mean flow is an effective process of stabilizing short shear waves. The numerical results we obtained (cf. table 1) also show that k_2 is positive for very small values of α. This implies that for extremely long waves, energy can actually be transferred from higher harmonics to the fundamental, an instability mechanism which was not found in shear waves. For relatively short gravity waves, however, k_2 are all negative which implies that energy can only be transferred from the fundamental to generate higher harmonics; this is the same stability mechanism conjectured by Stuart (1960) and confirmed by Reynolds & Potter (1967) for short shear waves in a parallel flow without a free surface. Turning to the sign of k_3, we found that for very long waves k_3 is negative due to the large rate of work done by the normal stress to deform the free surface. For relatively short waves all calculated values of k_3 are positive and then process (c) is destablizing. Along each curve of constant amplitude in figures 3 to 5, the sum of k_1, k_2 and k_3 is negative. For the portion of such a curve where α is sufficiently small process (c) dominates, and for the rest of the curve process (b) dominates.

Finally, the supercritically stable wave régimes shown in figure 1 and 2 are seen to rotate about the bifurcation point in a clockwise direction as the surface tension is increased. Clearly, this indicates that the surface tension has the sole role of stablizing the flow. Moreover, both neutral curves and curves of zero Landau second coefficient do not form a loop as they do in parallel shear flows. It follows that within the flow régime considered viscosity has only one role of stabilizing film flows through viscous dissipation.

4. Conclusion

A parallel flow with a free surface does not exhibit subcritical instability. Nor could it sustain a subcritically stable wave motion. All two-dimensional long gravity waves of finite but small amplitudes are damped in the subcritical region in the α, R plane. Supercritically stable wave régimes are found. For very long gravity waves, non-linear stability is found to be due to the large rate of work done by the normal stress to deform the free surface. For relatively short gravity waves, however, the stability is shown to be mainly due to the generation of higher harmonics which leads to the distortion of the wave form. Unlike the situation in parallel shear flows, the mechanism of non-linear stability by distortion of the mean flow through Reynolds stresses is found to be negligibly small. It should be pointed out that the above conclusion is valid only for long gravity waves. The non-linear stability mechanism for short capillary waves is likely to be more complicated due to the possibility of competition between gravity waves and shear waves for instability (Lin 1967).

REFERENCES

Benjamin, T. B. 1957 *J. Fluid Mech.* **2**, 544.

Binnie, A. M. 1957 *J. Fluid Mech.* **2**, 551.

Crapper, G. D. 1957 *J. Fluid Mech.* **2**, 532.

Kapitza, P. L. 1949 *Zh. Eksperim. i Teor. Fiz.* **19**, 105. (*Collected papers of P. L. Kapitza*, vol. II, p. 662. Oxford: Pergamon, 1964.)

LIN, S. P. 1967 *Phys. Fluids*, **10**, 308.
LIN, S. P. 1969 *J. Fluid Mech.* **36**, 113.
RALSTON, A. & WILF, H. S. 1967 *Mathematical Methods for Digital Computers*, vol. 2. New York: Wiley.
REYNOLDS, W. C. & POTTER, M. C. 1967 *J. Fluid Mech.* **27**, 465.
STUART, J. T. 1956 *J. Aero. Sci.* **23**, 86, 894.
STUART, J. T. 1960 *J. Fluid Mech.* **9**, 353.
STOKES, G. G. 1847 *Trans. Camb. Phil. Soc.* **8**, 441.
YIH, C. S. 1963 *Phys. Fluids*, **6**, 321.